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LETTER TO THE EDITOR

Lifetime of excitations in a clean Luttinger liquid

K V Samokhin†
Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CB3 0HE, UK

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Abstract. Non-linear band dispersion in the Luttinger model of one-dimensional interacting electrons gives rise to collisions between the bosonic sound-like excitations. The decay rate is calculated beyond a plain perturbation theory both at $T = 0$ and $T \neq 0$. A self-consistent approach reveals a non-analytical dependence of the lifetime of excitations on the coupling constant, wave vector and temperature.

The notion of a 'Luttinger liquid' was introduced [1] to describe the peculiar low-energy physics of interacting fermions in one dimension [2]. Unlike the Fermi liquids in higher dimensions, there are no single-particle excitations in 1D systems and their dynamics are essentially collective. The basic tool of Luttinger liquid theory is the bosonization technique [3], whose success is based on the possibility of linearization of the single-electron spectrum in the vicinity of Fermi points. The properties of correlated fermion systems are then described entirely in terms of free bosonic excitations with linear dispersion. The lifetime of these excitations becomes finite if we take into account extra interactions, e.g., with impurities or phonons. However, in a clean system without phonons the only contribution to the damping of excitations may come from their mutual collisions. It has long been realized [1] that such interactions originate from a non-zero band curvature, which is always present in real systems.

Previous work has been concentrated on calculation of the damping of collective excitations (1D plasmons) in quantum wires with Coulomb interaction, where the dispersion law is non-linear: $\omega(k) \sim k|\ln k|^{1/2}$ [4]. In the random-phase approximation (RPA) [5], the decay rate of plasmons is given by the imaginary part of the bare polarization operator (Landau damping). However, this imaginary part is zero at the mass shell for the sound-like excitations in the short-range case, as well as for 1D plasmons [6]. Therefore, in order to calculate the lifetime of excitations one should either allow for impurity scattering [7], or go beyond the RPA, the latter being equivalent to finding the corrections due to a non-zero band curvature. Such corrections to the RPA for 1D plasmons have been calculated within a conventional perturbation theory for realistic band spectra [4, 7, 8]. The disadvantage of such an approach is that one has to use approximations, the validity of which in 1D needs some justification itself. For this reason, we choose to resort to the bosonization approach, in which the band curvature effects can be taken into account explicitly by adding extra interaction terms to the bosonized Hamiltonian [1]. This allows one to treat all such corrections in a systematic way. However, the task of calculating the decay rate is far from trivial, since the perturbative analysis using the golden rule fails in the short-range case. The

† Permanent address: L D Landau Institute for Theoretical Physics, Kosygina Street 2, 117940 Moscow, Russia.

reason is that the laws of energy and momentum conservation are satisfied simultaneously for the waves with linear dispersion, so that one would obtain an unphysical infinite decay rate in the second-order perturbation theory. Therefore, a more accurate treatment is needed.

In this letter, we present a self-consistent calculation of the damping of the long-wavelength excitations in a clean Luttinger liquid. To be specific, let us consider a 1D system of spinless fermions with short-range interaction, described by the Luttinger model [9]. The basic feature of this model is that the single-electron spectrum consists of two branches—right (R) and left (L) 'movers'—both with linear dispersion and unconstrained momentum and energy. The Hamiltonian is $H = H_0 + H'$, where

$$H_0 = \mathrm{i}v_\mathrm{F}\int_0^L \mathrm{d}x\,\left(\psi_R^\dagger(x)\partial_x\psi_R(x) - \psi_L^\dagger(x)\partial_x\psi_L(x)\right) + \frac{1}{2}\int_0^L \mathrm{d}x\,\mathrm{d}x'\,\rho(x)U(x-x')\rho(x')$$
$$H' = -\frac{1}{2m}\int_0^L \mathrm{d}x\,\left(\psi_R^\dagger(x)\partial_x^2\psi_R(x) + \psi_L^\dagger(x)\partial_x^2\psi_L(x)\right). \tag{1}$$

Here $\rho(x) = \psi_R^\dagger(x)\psi_R(x) + \psi_L^\dagger(x)\psi_L(x)$ is the electron density operator; $U(x)$ has a finite range R in real space, its Fourier transform $U(k)$ being equal to U_0 at $kR \ll 1$. The second term in the Hamiltonian takes account of a non-zero curvature of the single-electron spectrum $\epsilon_p = v_\mathrm{F}(|p| - p_\mathrm{F}) + (|p| - p_\mathrm{F})^2/2m$ near the Fermi points [1]. The fermion operators can be written in the bosonized form: $\psi_{R(L)}(x) \sim \mathrm{e}^{\pm\mathrm{i}p_\mathrm{F}x - \mathrm{i}\Phi_{R(L)}(x)}$ [1, 10], where

$$\Phi_{R(L)}(x) = \pm\frac{\pi x}{L}N_{R(L)} - \mathrm{i}\sum_{k\neq 0}\alpha(\pm k)\left(\mathrm{e}^{\mathrm{i}kx}b_k^\dagger - \mathrm{e}^{-\mathrm{i}kx}b_k\right). \tag{2}$$

Here $\alpha(k) = (2\pi/L|k|)^{1/2}(\theta(k)\cosh\varphi - \theta(-k)\sinh\varphi)$ ($L \to \infty$), $\theta(x)$ is the step function, and $\mathrm{e}^{-4\varphi} = 1 + 2U_0/\pi v_\mathrm{F}$. The operators $N_{R(L)}$ correspond to the number of particles added to the right (left) branch of spectrum with respect to the ground state density $N/L = p_\mathrm{F}/\pi$. The operators $b_k^\dagger$, b_k create and annihilate non-uniform excitations and obey canonical boson commutation relations. The Hamiltonian H_0 then describes free bosonic excitations with linear dispersion $\omega(k) = v|k|$, where $v = v_\mathrm{F}e^{-2\varphi}$.

The bosonized form of H' looks as follows [1]:

$$H' = \frac{1}{12\pi m}\int \mathrm{d}x\,\left(:(\partial_x\Phi_R)^3: - :(\partial_x\Phi_L)^3:\right) \tag{3}$$

where the colons mean boson normal ordering. After substitution of (2) in (3), we obtain the Hamiltonian

$$H' = \int\frac{\mathrm{d}k_1}{2\pi}\frac{\mathrm{d}k_2}{2\pi}\frac{\mathrm{d}k_3}{2\pi}\,V(k_1,k_2,k_3)\Big\{\Big(b_{k_1}b_{k_2}^\dagger b_{k_3}^\dagger\delta(k_1-k_2-k_3)$$
$$+\ (\text{all permutations of } k_1,k_2,k_3)\Big) + b_{k_1}b_{k_2}b_{k_3}\delta(k_1+k_2+k_3) + \text{h.c.}\Big\} \tag{4}$$

which describes triple collisions between excitations. The vertex is

$$V(k_1,k_2,k_3) = \frac{1}{6m}\frac{k_1k_2k_3}{|k_1k_2k_3|^{1/2}}\Big\{\prod_i\Big(\theta(k_i)\cosh\varphi - \theta(-k_i)\sinh\varphi\Big) - (k_i \to -k_i)\Big\}$$
$$(i = 1,2,3). \tag{5}$$

We restrict our analysis to the case of $N_R = N_L = 0$. However, as it is pointed out in [1], the state $|N_R = N_L = 0\rangle$ is not the ground one, since the Hamiltonian of the uniform density fluctuations is unstable. In order to guarantee stability, one has to take into account the fourth-order interaction originating from the next term in the expansion of the single-electron spectrum, which has the form

$$H_{\rm int}^{(4)} = (\beta/96\pi m^2 v_{\rm F}) \int {\rm d}x\{: (\partial_x \Phi_R)^4 : - : (\partial_x \Phi_L)^4 :\}.$$

At $\beta > 3/4$, the ground state is $|N_R = N_L = 0\rangle$ [1]. We assume that the conditions of stability are satisfied. Since the fourth-order vertex contains higher powers of k, its contribution to the decay rate is small compared with that from the cubic interaction.

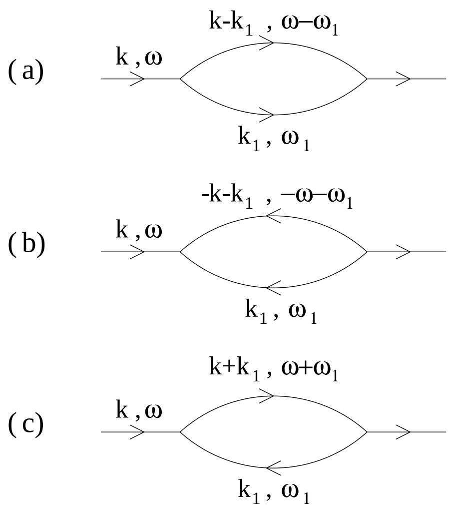

Figure 1. Diagrams for the self-energy function corresponding to three different channels of damping bosonic excitations: (a) spontaneous decay, (b) three-wave annihilation and (c) absorbtion of another excitation. Arrowed lines correspond to the Green functions of Luttinger bosons.

We are now in a position to calculate the damping of interacting excitations by the standard means of quantum field theory. The exact thermodynamic Green function of Luttinger bosons has the form:

$$G(k, \omega_n) = \frac{1}{{\rm i}\omega_n - v|k| - \Sigma(k, \omega_n)} \qquad (\omega_n = 2\pi n T). \tag{6}$$

The self-energy function Σ is given by the sum of the three diagrams in figure 1. After analytical continuation $\Sigma(k, \omega_n) \to \Sigma^R(k, \omega) = -{\rm i}\Gamma_{k,\omega}$ ($\Gamma > 0$), we obtain a self-consistent equation for the function Γ:

$$\begin{aligned} \Gamma_{k,\omega} = {} & \\ 18\int_{-\infty}^{\infty} \frac{{\rm d}k_1}{2\pi}\frac{{\rm d}\omega_1}{2\pi}\Big\{ & V^2(k, k_1, k-k_1)\left(\coth\frac{\omega_1}{2T} + \coth\frac{\omega-\omega_1}{2T}\right) {\rm Im}\, G^R(k_1, \omega_1) {\rm Im}\, G^R(k-k_1, \omega-\omega_1) \\ & - V^2(k, k_1, -k-k_1)\left(\coth\frac{\omega_1}{2T} - \coth\frac{\omega+\omega_1}{2T}\right) {\rm Im}\, G^R(k_1, \omega_1) {\rm Im}\, G^R(-k-k_1, -\omega-\omega_1) \\ & + 2V^2(k, k_1, k+k_1)\left(\coth\frac{\omega_1}{2T} - \coth\frac{\omega+\omega_1}{2T}\right) {\rm Im}\, G^R(k_1, \omega_1) {\rm Im}\, G^R(k+k_1, \omega+\omega_1)\Big\} \end{aligned} \tag{7}$$

where

$$ {\rm Im}\, G^R(k, \omega) = -\frac{\Gamma_{k,\omega}}{(\omega - v|k|)^2 + \Gamma_{k,\omega}^2} \ . \tag{8}$$

We are interested in calculation of the damping of bosonic excitations at the mass shell, which is denoted as $\gamma_k = \Gamma_{k,\omega=v|k|}$. In the second-order perturbation theory this quantity is infinite. Indeed, if we substitute in equation (7) the bare Green functions instead of the exact ones, which amounts to replacing $\mathrm{Im}\, G_0^R(k,\omega) \to -\pi\delta(\omega - v|k|)$, then $\Gamma^{(0)}_{k,\omega} \sim \delta(\omega - v|k|)$ and $\gamma_k = \infty$. Thus, we have to solve the problem self-consistently. One might expect that as a result of self-consistent treatment the δ-function singularity is rounded off, so that we assume the following form of solution:

$$\Gamma_{k,\omega} = \gamma_k f\left(\frac{\Omega}{\gamma_k}\right) \tag{9}$$

where $\Omega = \omega - v|k|$. Function $f(x)$ has a peak near $x = 0$ ($f(0) = 1$) and quickly vanishes at $x \to \infty$.

We start with the case of $T = 0$. Let $k > 0$, then we split the range of integration over k_1 on the right-hand side of equation (7) and retain only the most singular contributions, whose denominators go to zero fastest of all (as $(\omega_1 - vk_1)^4$) when $\omega = vk$ and $\Gamma \to 0$. The result is

$$\gamma_k = 36\int_0^k \frac{\mathrm{d}k_1}{2\pi}\int_0^{vk}\frac{\mathrm{d}\omega_1}{2\pi}\, V^2(k,k_1,k-k_1)\frac{\Gamma_{k_1,\omega_1}}{(\omega_1 - vk_1)^2 + \Gamma^2_{k_1,\omega_1}}\,\frac{\Gamma_{k-k_1,vk-\omega_1}}{(\omega_1 - vk_1)^2 + \Gamma^2_{k-k_1,vk-\omega_1}}$$
$$-72\int_0^\infty \frac{\mathrm{d}k_1}{2\pi}\int_{-vk}^{0}\frac{\mathrm{d}\omega_1}{2\pi}\, V^2(k,k_1,k+k_1)\frac{\Gamma_{k_1,\omega_1}}{(\omega_1 - vk_1)^2 + \Gamma^2_{k_1,\omega_1}}\,\frac{\Gamma_{k+k_1,vk+\omega_1}}{(\omega_1 - vk_1)^2 + \Gamma^2_{k+k_1,vk+\omega_1}}\,.$$

The function of ω_1 in the integrand has a sharp peak near $\omega_1 = vk_1$, which is situated outside the range of integration over ω_1 in the second term, so that this term can be safely neglected. This fact has a simple physical interpretation: at $T = 0$ there is no excitation in the system, so that the only contribution to the decay rate is due to spontaneous decay (see figure 1(a)). In the first term, it is possible to extend the limits of integration over ω_1 to infinity. Using (9), we obtain:

$$\gamma_k = 36\int_0^k \frac{\mathrm{d}k_1}{2\pi}\, V^2(k,k_1,k-k_1)\int_{-\infty}^{\infty}\frac{\mathrm{d}\Omega_1}{2\pi}\,\frac{\gamma_{k_1} f\left(\Omega_1/\gamma_{k_1}\right)}{\Omega_1^2 + \gamma_{k_1}^2 f^2\left(\Omega_1/\gamma_{k_1}\right)}\,\frac{\gamma_{k-k_1} f\left(\Omega_1/\gamma_{k-k_1}\right)}{\Omega_1^2 + \gamma_{k-k_1}^2 f^2\left(\Omega_1/\gamma_{k-k_1}\right)} \tag{10}$$

where $\Omega_1 = \omega_1 - vk_1$. We seek a solution of equation (10) in the form $\gamma_k = A|k|^\alpha$. Substituting (5) and introducing new variables $x = k_1/k$ and $y = \Omega_1/\gamma_{k_1}$, we obtain $A^2k^{2\alpha} = \lambda^2 k^4 I_1(\alpha)$ where

$$\lambda = \frac{1}{2\pi m}\left(1 + \frac{2U_0}{\pi v_{\mathrm{F}}}\right)^{-3/4}\left(1 + \frac{3U_0}{2\pi v_{\mathrm{F}}}\right) \tag{11}$$

and

$$I_1(\alpha) = \int_0^1 \frac{x\,\mathrm{d}x}{(1-x)^{\alpha-1}}\int_{-\infty}^{\infty}\mathrm{d}y\,\frac{f(y)}{y^2 + f^2(y)}\,\frac{f(Y)}{Y^2 + f^2(Y)}$$

$Y = y(x/1-x)^\alpha$. Therefore, $\alpha = 2$ and, finally

$$\gamma_k(T = 0) = c_1|\lambda|k^2 \tag{12}$$

where $c_1 = I_1^{1/2}(2)$ is a numerical coefficient, which can be calculated only if the whole function $f(x)$ is known. However, the dependence of γ_k on the wave number, which is of the most interest for us, is insensitive to the precise form of $f(x)$. Note that, as seen from (5) and (11), Luttinger bosons interact with each other and, therefore, have a finite lifetime at $U_0 = 0$. The reason is that they are not exact eigenstates of the system, unless the single-electron spectrum is linear. Indeed, since the operators $b_k^\dagger, b_k$ are directly related to the right and left fermion density operators $\rho_{R(L)}(k) = \sum_p c^\dagger_{R(L),p+k} c_{R(L),p}$ [1, 2], the bosonic excitations are linear combinations of electron–hole pairs with total momentum k and energy $\varepsilon_p(k) = \epsilon_{p+k} - \epsilon_p$. In contrast to the linearized case, where $\varepsilon_p(k) = v_{\mathrm{F}} k$, in the case of non-linear band dispersion $\varepsilon_p(k)$ depends on p, so that the excited states created by $b_k^\dagger$ are made up of pairs with different energies and thus cannot be eigenstates of the non-interacting Hamiltonian.

At $T \neq 0$, the main contributions come from the long-wavelength excitations, so that one can replace the Bose–Einstein functions by the Rayleigh–Jeans distribution: $\coth(\omega/2T) \to 2T/\omega$ (it is assumed that $vk \ll T$). Proceeding as before, we retain only the most singular at $\omega = vk$ and $\Gamma \to 0$ terms on the right-hand side of (7), replace $\omega_1 \to vk_1$ in the arguments of the distribution functions, assume that $\gamma_k = A|k|^\alpha$, and end up with the equation $A^2 k^{2\alpha} = \lambda^2 (T/v) k^3 I_2(\alpha)$, where λ is given by (11) and

$$I_2(\alpha) = \int_0^1 \frac{\mathrm{d}x}{(1-x)^\alpha} \int_{-\infty}^{\infty} \mathrm{d}y \frac{f(y)}{y^2 + f^2(y)} \frac{f(Y_-)}{Y_-^2 + f^2(Y_-)} + 2\int_0^\infty \frac{\mathrm{d}x}{(1+x)^\alpha} \times \int_{-\infty}^{\infty} \mathrm{d}y \frac{f(y)}{y^2 + f^2(y)} \frac{f(Y_+)}{Y_+^2 + f^2(Y_+)}$$

$Y_\pm = y(x/1 \pm x)^\alpha$. Therefore, $\alpha = 3/2$ so that

$$\gamma_k(T) = c_2|\lambda| \sqrt{\frac{T}{v}}\ |k|^{3/2} \tag{13}$$

where $c_2 = I_2^{1/2}(3/2)$. Thus, the decay rate is found to depend non-analytically on the coupling constant, temperature and wave number. At $vk \ll T$ the long-wavelength excitations in Luttinger liquid can be considered as classical waves. Since the k-dependence of equation (5) is the same as that of the phonon interaction vertex in conventional hydrodynamics [11], it is not surprising that the decay rate of Luttinger bosons has the same T- and k-dependences as the sound attenuation in 1D classical liquids [12].

In the case of long-range interaction, the non-linearity of the dispersion of 1D plasmons prevents the laws of energy and momentum conservation from being satisfied simultaneously. Although the decay rate is zero in the second-order perturbation theory, it is not in higher orders, so that we do not expect any peculiar behaviour in this case. As for the contribution to γ_k from impurity scattering, it is proportional to k^2 and does not depend on temperature [13]. In real situations, the finite-size effects should also be taken into account. In the case of spin-1/2 fermions, it turns out that the spin–charge separation [2] is broken down by a non-zero band curvature, so that the spin and charge density waves interact with each other and acquire a finite lifetime. This case will be considered in a separate publication [14].

A non-analytical behaviour of the decay rate would manifest itself, for example, in a peculiar T- or k-dependence of the peaks in the dynamic structure factor $S(k,\omega) = 2(1 - \mathrm{e}^{-\omega/T})^{-1} \mathrm{Im}\, K^R(k,\omega)$ [15], which can be measured directly in Raman scattering experiments. The density correlation function K^R is

$$K^R(k,\omega) = \frac{v_{\mathrm{F}}}{\pi} \frac{k^2}{v^2k^2 - (\omega + \mathrm{i}\Gamma_{k,\omega})^2}. \tag{14}$$

The peaks in $S(k,\omega)$ at $\omega = v|k|$ have a finite width of the order of $\gamma_k(T)$ with magnitude proportional to $T^{1/2}k^{-3/2}$. Although the results of, e.g., inelastic light-scattering measurements in GaAs quantum wires [16] agree with the predictions of the perturbative RPA-based approach, one cannot rule out unambiguously the possibility of a non-analytical dependence of the peak intensity on the basis of available experimental data. It is an open question, whether 1D electrons in real quantum wires form a Luttinger liquid, or can still be described within the Fermi liquid theory. Experimental discovery of the peculiar behaviour of the damping of collective excitations would be strong evidence in favour of the Luttinger liquid picture.

To summarize, we have presented a self-consistent calculation of the lifetime of bosonic excitations in a clean Luttinger liquid with short-range interaction, the most noticeable feature being its non-analytical dependence on the coupling constant (at $T = 0$) and the wave number and temperature (at $T \neq 0$).

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